**SYDNEY BOYS HIGH SCHOOL**
**MOORE PARK, SURRY HILLS**

**2014**
**HIGHER SCHOOL CERTIFICATE**
**ASSESSMENT TASK #1**

# Mathematics Extension 2

**General Instructions**

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each question is to be returned in a separate bundle.
- All necessary working should be shown in every question.

**Total Marks – 60**

- Attempt questions 1 – 3
- All questions are of equal value.
- Unless otherwise directed give your answers in simplest exact form.

Examiner: *A.M.Gainford*

**Question 1. (Start a new page.)** (20 marks)

**Marks**

(a) For the complex number $z = 1 - \sqrt{3}i$ find: **3**

(i) $|z|$

(ii) arg $z$.

(iii) $\frac{z}{i}$

(b) Express the following in the form $a + ib$ (for real $a$ and $b$). **2**

(i) $(6+5i)\overline{(4-i)}$

(ii) $\frac{-2+3i}{3-4i}$

(c) Find the square roots of $9+40i$, giving your answers in the form $x + iy$. **2**

**Question 1 continues on the next page.**

(d) Sketch (on separate diagrams) the region in the Argand diagram containing the points z for which: **4**

(i) $\frac{\pi}{4} \leq \arg(z) \leq \frac{\pi}{2}$ and $|z-1-3i| \leq 2$

(ii) $\arg\left(\frac{z-2i}{z+2}\right) = \frac{\pi}{4}$

(e) (i) Express 1 + *i* in modulus-argument form. **1**

(ii) Given that $(1+i)^n = x+iy,$ where *x* and *y* are real, and *n* is an integer, prove that $x^2 + y^2 = 2^n$ **2**

(f) Which complex numbers are the reciprocals of their conjugates? **1**

(g) Consider the function $y = 2\cos^{-1}(x^2 - 1)$. **5**

(i) Determine the domain and range of the function.

(ii) Sketch the graph of the function showing important features.

(iii) Find the derivative of the function and state the values of *x* for which it is defined.

**Question 2. (Start a new page.)** (20 marks)

**Marks**

(a) The points *O*, *I*, *Z*, and *P* on the Argand diagram represent the complex numbers 0, 1, $z$, and $z+1$ respectively, where $z=\cos\theta+i\sin\theta$ is any complex number of modulus 1, and $0<\theta<\pi$. **4**

(i) Explain why *OIPZ* is a rhombus.

(ii) Show that $\dfrac{z-1}{z+1}$ is purely imaginary.

(iii) Find the modulus of $z+1$ in terms of $\theta$.

(b) Differentiate $x\sin 2x$, and hence find $\int x\cos 2x\ dx$. **2**

(c) Given that $2-i$ is a root of the equation $x^4-6x^3+10x^2+2x-15=0$: **5**

(i) state another complex (non-real) root, giving a reason.

(ii) find all roots of the equation.

(iii) write the equation in fully factored form over the complex field.

(d) Consider the functions $y=-\cos^{-1}\left(\frac{x}{2}\right)$ and $y=\frac{1}{2}\tan^{-1}(x)-\frac{\pi}{2}$. **4**

(i) Show that the graphs of these functions intersect on the *y*-axis.

(ii) Show that the graphs have a common tangent at the point of intersection, and write the equation of this tangent.

(e) Given the quadratic equation $x^2-x-3=0$ with roots $\alpha_1, \alpha_2$ : **5**

(i) Show that $x^4=7x+12$.

(ii) Hence or otherwise find a quadratic equation with roots $\alpha_1^4$ and $\alpha_2^4$.

**Question 3. (Start a new page.)** (20 marks)

**Marks**

(a) (i) Find the five roots of the equation $z^5 = 1$. Give the roots in modulus-argument form. **2**

(ii) Show that $z^5 - 1$ can be factorised in the form : **2**

$$z^5 - 1 = (z-1)(z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$$

(iii) Hence show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$, and hence find the exact value of $\cos\frac{2\pi}{5}$. **3**

(b) When a polynomial $P(x)$ is divided by $x-2$ and by $x-3$ the remainders are 4 and 9 respectively. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$. **2**

(c) Ten people, consisting of three couples and four singles are to be seated randomly at a round table. **3**

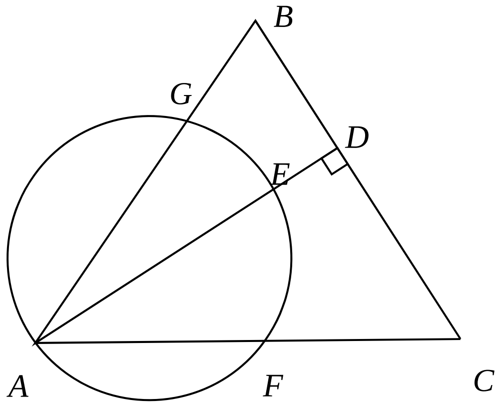
(i) How many arrangements are possible?

(ii) What is the probability (as a simplified fraction) that all three couples are seated as couples, separated from other couples by one or two singles?

(d) Prove that the polynomial equation $ax^4 + bx + c = 0$ , where *a, b,* and c are non-zero, cannot have a triple root. **1**

(e) Use the substitution $x = 2\sin\theta$ , or otherwise, to evaluate $\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx$ . **3**

(f) In the triangle $ABC$, $AD$ is the perpendicular from $A$ to $BC$. The point $E$ is any point on $AD$, and the circle drawn with $AE$ as diameter cuts $AC$ at $F$ and $AB$ at $G$. **4**




(i) Copy the diagram to your answer booklet.

(ii) Prove that $B$, $G$, $F$, and $C$ are concyclic.

**This is the end of the paper.**

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## STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

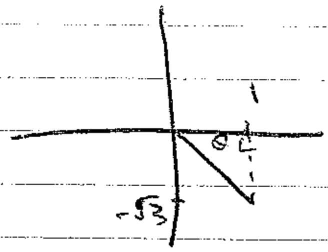
$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

1) a) i) $z = 1 - \sqrt{3}i$




$|z| = \sqrt{(1)^2 + (-\sqrt{3})^2}$

$= 2$

ii) $\arg z = -\frac{\pi}{3}$

iii) $\frac{1-\sqrt{3}i}{i} \times \frac{-i}{-i} = \frac{-i - \sqrt{3}}{1}$

$= -\sqrt{3} - i$

b) i) $(6+5i)\overline{(4-i)} = (6+5i)(4+i)$

$= 24 + 6i + 20i - 5$

$= 19 + 26i$

ii) $\frac{-2+3i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{-6-8i+9i-12}{9+16}$

$= \frac{-18+i}{25}$

$= -\frac{18}{25} + \frac{1}{25}i$

c) $(x+iy)^2 = 9 + 40i$

$x^2 - y^2 + 2xyi = 9 + 40i$

equate

$x^2 - y^2 = 9$ ——— ①

$2xy = 40$ ——— ②

$(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2$

$= 81 + 1600$

$= 1681$

$x^2 + y^2 = 41$ ——— ③

① + ③

$2x^2 = 50$

$x^2 = 25$

$x = \pm 5$

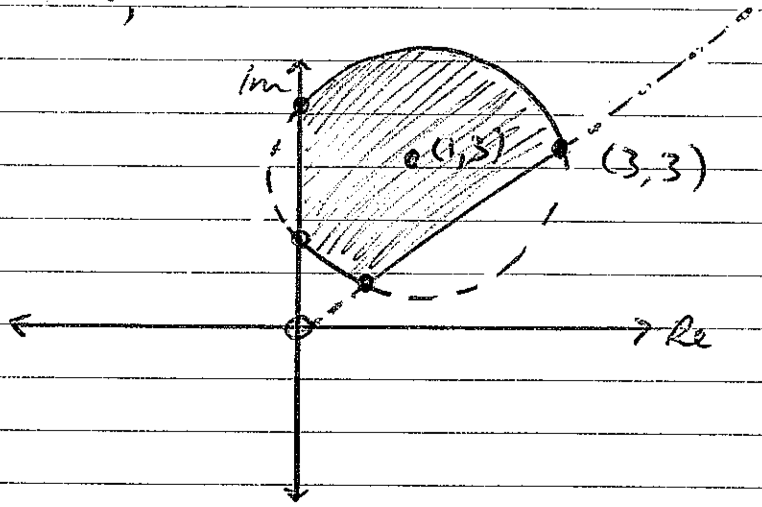
sub into ②

$2(\pm 5)y = 40$

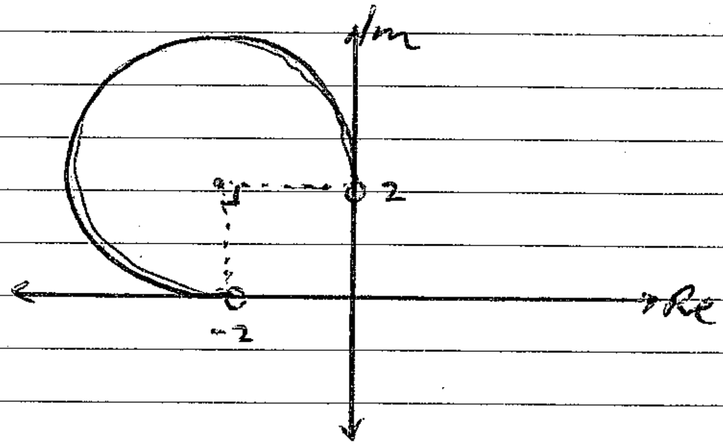
$y = \pm 4$

$\therefore\ 5+4i,\ -5-4i$

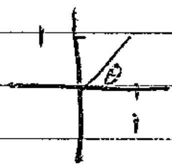
d) i)




ii)




e) i) $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$




ii) $(1+i)^n = x+iy$

$\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^n = x+iy$

$\left(2^{\frac{1}{2}}\right)^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = x+iy$

$\left|2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\right| = |x+iy|$

$2^{\frac{n}{2}} = \sqrt{x^2+y^2}$

$x^2+y^2 = 2^n$

f) complex numbers with a modulus of 1.

consider $z = \frac{1}{\bar{z}}$

$z\bar{z} = 1$

$|z|^2 = 1$

$|z| = 1$

g) $y = 2\cos^{-1}(x^2-1)$

i) D: $-1 \leq x^2 - 1 \leq 1$

$0 \leq x^2 \leq 2$

$-\sqrt{2} \leq x \leq \sqrt{2}$

R: $x^2 \geq 0$

$x^2 - 1 \geq -1$

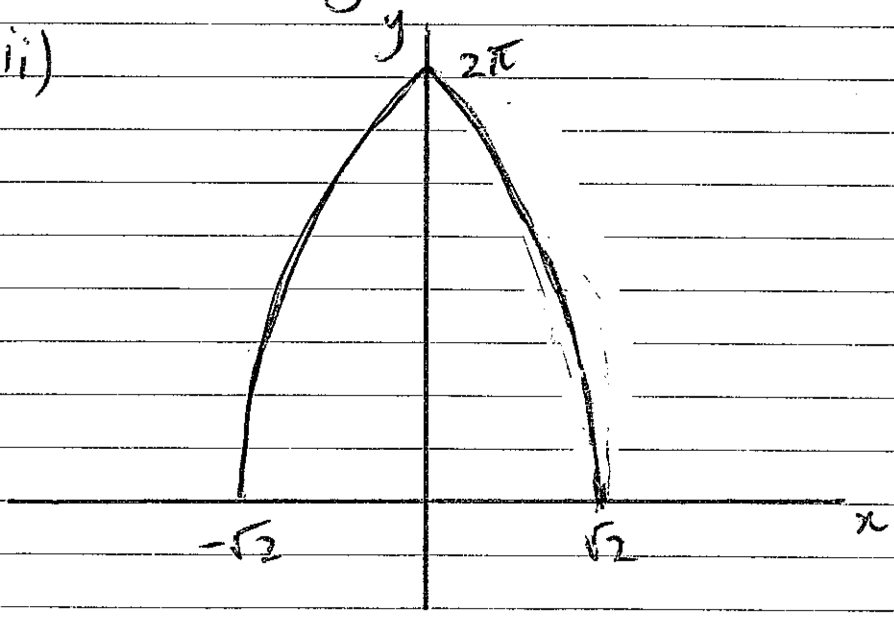
since $x^2-1$ will give all values between -1 & 1

$0 \leq \cos^{-1}(x^2-1) \leq \pi$

$0 \leq 2\cos^{-1}(x^2-1) \leq 2\pi$

$0 \leq y \leq 2\pi$

ii)



iii) $y = 2\cos^{-1}(x^2-1)$

$$y' = 2 \times \frac{-1}{\sqrt{1-(x^2-1)^2}} \times 2x$$

$$= \frac{-4x}{\sqrt{1-(x^4-2x^2+1)}}$$

$$= \frac{-4x}{\sqrt{2x^2-x^4}}$$

OR

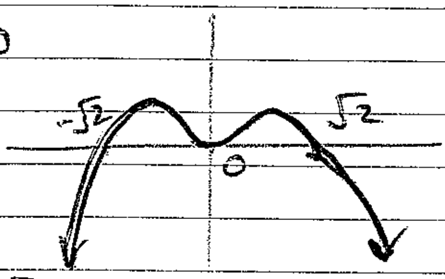
$$= \begin{cases} \dfrac{4}{\sqrt{2-x^2}}, & \text{when } x > 0 \\ \dfrac{4}{\sqrt{2-x^2}}, & \text{when } x < 0 \end{cases}$$

The derivative is defined when

$2x^2 - x^4 > 0$

$x^2(2-x^2) > 0$

$x^2(\sqrt{2}-x)(\sqrt{2}+x) > 0$




$-\sqrt{2} < x < 0, \ 0 < x < \sqrt{2}$

OR

$-\sqrt{2} < x < \sqrt{2}, \ x \neq 0$

<u>Note:</u> as $x \to 0^+$, $y' \to -2\sqrt{2}$

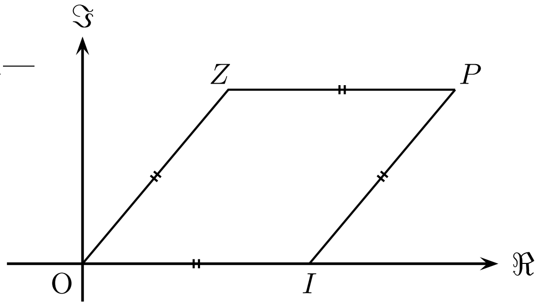
as $x \to 0^-$, $y' \to 2\sqrt{2}$

2014 Extension 2 Mathematics Task 1:
**Solutions— Question 2**

2. (a) The points $O$, $I$, $Z$, and $P$ on the Argand diagram represent the complex numbers 0, 1, $z$, and $z+1$ respectively, where $z = \cos\theta + i\sin\theta$ is any complex number of modulus 1, and $0 < \theta < \pi$. 4

(i) Explain why $OIPZ$ is a rhombus.

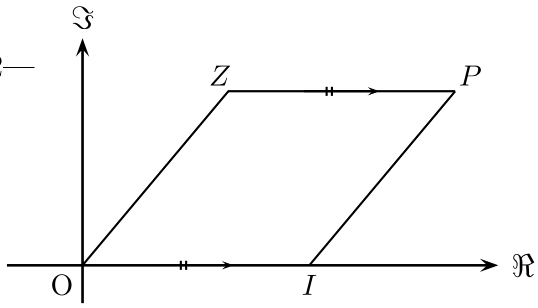
**Solution:** Method 1—




$$|OI| = 1, \quad |OZ| = 1,$$
$$|ZP| = |z+1-z| = |1| = 1. \quad |IP| = |z+1-1| = |z| = 1.$$

$\therefore$ $OIPZ$ is a rhombus (equal sides).

**Solution:** Method 2—




$|OI| = |ZP| = 1$ by construction,
$OI \parallel ZP$,
$\therefore$ $OIPZ$ is a parallelogram (opp. sides equal and parallel),
$|OI| = |OZ| = 1$ (given),
$\therefore$ $OIPZ$ is a rhombus.

(ii) Show that $\dfrac{z-1}{z+1}$ is purely imaginary.

**Solution:** Method 1—
Consider the diagonals of the rhombus $OIPZ$:

$$\begin{aligned} OP &= z+1, \\ IZ &= z-1, \\ \arg(z-1) - \arg(z+1) &= \frac{\pi}{2}, \ (OP \perp IZ, \text{ diagonals of rhombus}) \\ \textit{i.e.}, \ \arg\left(\frac{z-1}{z+1}\right) &= \frac{\pi}{2}. \end{aligned}$$

So $\dfrac{z-1}{z+1}$ must lie on the imaginary axis and is purely imaginary.

**Solution:** Method 2—

$$\begin{aligned}\frac{z-1}{z+1}\times\frac{\overline{z}+1}{\overline{z}+1} &= \frac{z\overline{z}+z-\overline{z}-1}{z\overline{z}+z+\overline{z}+1},\\ &= \frac{1+2i\sin\theta-1}{1+2\cos\theta+1},\\ &= \frac{2i\sin\theta}{2+2\cos\theta},\\ &= \frac{i\sin\theta}{1+\cos\theta},\ \text{which is purely imaginary.}\end{aligned}$$

**Solution:** Method 3—

If $\dfrac{z-1}{z+1}$ is purely imaginary, then $\dfrac{z-1}{z+1}+\overline{\left(\dfrac{z-1}{z+1}\right)}=0$.

$$\begin{aligned}\text{L.H.S.} &= \frac{z-1}{z+1}+\frac{\overline{z}-1}{\overline{z}+1},\\ &= \frac{z\overline{z}+z-\overline{z}-1+z\overline{z}+\overline{z}-z-1}{z\overline{z}+z+\overline{z}+1}.\\ \text{But } z\overline{z} &= |z|^2=1,\\ \text{so L.H.S.} &= \frac{0}{z+\overline{z}+2},\\ &= 0,\\ &= \text{R.H.S.}\end{aligned}$$

**Solution:** Method 4—

$$\frac{\cos\theta+i\sin\theta-1}{\cos\theta+i\sin\theta+1}\times\frac{\cos\theta-i\sin\theta+1}{\cos\theta-i\sin\theta+1}$$

$$= \frac{\cos^2\theta-i\sin\theta\cos\theta+\cos\theta+i\sin\theta\cos\theta+\sin^2\theta+i\sin\theta-\cos\theta+i\sin\theta-1}{\cos^2\theta-i\sin\theta\cos\theta+\cos\theta+i\sin\theta\cos\theta+\sin^2\theta+i\sin\theta+\cos\theta-i\sin\theta+1}$$

$$= \frac{2i\sin\theta}{2+2\cos\theta},$$

$$= \frac{i\sin\theta}{1+\cos\theta},\ \text{which is purely imaginary.}$$

**Solution:** Method 5—

$$\begin{aligned}\frac{x-1+iy}{x+1+iy}\times\frac{x+1-iy}{x+1-iy} &= \frac{x^2+x-ixy-x-1+iy+ixy+iy+y^2}{(x+1)^2+y^2},\\ &= \frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}.\\ \text{But } x^2+y^2 &= 1\ (i.e.\ |z|^2),\\ \text{so } \frac{z-1}{z+1} &= \frac{2iy}{(x+1)^2+y^2},\ \text{which is purely imaginary.}\end{aligned}$$

**Solution:** Method 6—

$$\begin{aligned}\frac{z-1}{z+1} &= \frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1},\\ &= \frac{1 - 2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} - 1}{2\cos^2\frac{\theta}{2} - 1 + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} + 1},\\ &= \frac{-2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)},\\ &= \frac{i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)} \text{ (as } -1 = i^2),\\ &= i\tan\frac{\theta}{2} \text{ which is purely imaginary.}\end{aligned}$$

(iii) Find the modulus of $z+1$ in terms of $\theta$.

**Solution:**

$$\begin{aligned}|z+1|^2 &= (z+1)(\overline{z}+1),\\ &= 2 + 2\cos\theta \text{ as above,}\\ \therefore\ |z+1| &= \sqrt{2(1+\cos\theta)},\\ &= \sqrt{2\times 2\cos^2\frac{\theta}{2}},\\ &= 2\cos\frac{\theta}{2}.\end{aligned}$$

(b) Differentiate $x\sin 2x$, and hence find $\int x\cos 2x\,dx$. 2

**Solution:**

$$\begin{aligned}\frac{d}{dx}(x\sin 2x) &= \sin 2x + 2x\cos 2x,\\ i.e.,\ 2x\cos 2x &= \frac{d}{dx}(x\sin 2x) - \sin 2x.\\ \int 2x\cos 2x\,dx &= x\sin 2x - \int \sin 2x\,dx,\\ &= x\sin 2x + \frac{\cos 2x}{2} + C.\\ \text{So } \int x\cos 2x\,dx &= \frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + C.\\ \text{Alternatively, } \int 2x\cos 2x\,dx &= x\sin 2x - \int 2\sin x\cos x\,dx,\\ &= x\sin 2x - \sin^2 x + C.\\ \text{So } \int x\cos 2x\,dx &= \frac{x\sin 2x - \sin^2 x}{2} + C.\end{aligned}$$

(c) Given that $2-i$ is a root of the equation $x^4 - 6x^3 + 10x^2 + 2x - 15 = 0$: 5

(i) state another complex (non-real) root, giving a reason.

**Solution:** $2+i$, as polynomials with real coefficients have their complex roots occurring in conjugate pairs.

(ii) find all the roots of the equation.

**Solution:** Method 1—

Possible other roots are $\pm 1$, $\pm 3$, $\pm 5$.

$$\begin{aligned} P(1) &= 1 - 6 + 10 + 2 - 15, \\ &\neq 0. \\ P(-1) &= 1 + 6 + 10 - 2 - 15, \\ &= 0. \\ P(3) &= 81 - 162 + 90 + 6 - 15, \\ &= 0. \end{aligned}$$

$\therefore$ The roots are $2 \pm i$, $-1$, and $3$.

**Solution:** Method 2—

$$\begin{aligned} (x - 2 - i)(x - 2 + i) &= x^2 - 4x + 4 + 1, \\ &= x^2 - 4x + 5. \end{aligned}$$

$$\begin{array}{r|l} & x^2 \quad - 2x \quad - 3 \\ \hline x^2 - 4x + 5 & x^4 - 6x^3 + 10x^2 + 2x - 15 \\ & -x^4 + 4x^3 - 5x^2 \\ \hline & -2x^3 + 5x^2 + 2x \\ & 2x^3 - 8x^2 + 10x \\ \hline & -3x^2 + 12x - 15 \\ & 3x^2 - 12x + 15 \\ \hline & 0 \end{array}$$

$x^2 - 2x - 3 = (x - 3)(x + 1)$

$\therefore$ The roots are $2 \pm i$, $-1$, and $3$.

(iii) write the equation in fully factored form over the complex field.

**Solution:** $(x + 1)(x - 3)(x - 2 - i)(x - 2 + 1) = 0.$

(d) Consider the functions $y = -\cos^{-1}\left(\frac{x}{2}\right)$ and $y = \frac{1}{2}\tan^{-1}(x) - \frac{\pi}{2}$. 4

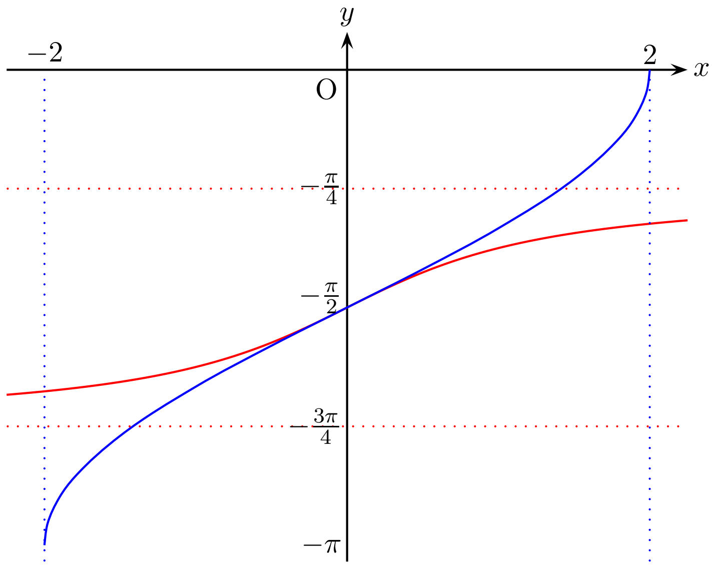
(i) Show that the graphs of these functions intersect on the $y$-axis.

**Solution:** For $y = -\cos^{-1}\left(\frac{x}{2}\right)$,

$$\begin{aligned} \text{Domain}: \quad & -1 \leqslant \tfrac{x}{2} \leqslant 1, \\ & -2 \leqslant x \leqslant 2. \\ \text{Range}: \quad & -\pi \leqslant y \leqslant 0. \\ \text{When} \quad & y = 0, \quad x = -\frac{\pi}{2}. \end{aligned}$$

For $y = \frac{1}{2}\tan^{-1}(x) - \frac{\pi}{2}$,

$$\begin{aligned} \text{Domain}: \quad & x \in \mathbb{R}, \\ \text{Range}: \quad & -\tfrac{\pi}{4} - \tfrac{\pi}{2} < y < \tfrac{\pi}{4} - \tfrac{\pi}{2}, \\ & -\tfrac{3\pi}{4} < y < -\tfrac{\pi}{4}, \\ \text{When} \quad & y = 0, \quad x = -\frac{\pi}{2}. \end{aligned}$$




From the common point $(0, -\frac{\pi}{2})$ and the sketch, it is clear that the curves have their intersection on the $y$-axis.

(ii) Show that these graphs have a common tangent at the point of intersection, and write the equation of this tangent.

**Solution:**

$$y = -\cos^{-1}\left(\frac{x}{2}\right), \qquad y = \frac{1}{2}\tan^{-1}(x) - \frac{\pi}{2},$$

$$\frac{dy}{dx} = -\frac{1}{2} \times \frac{-1}{\sqrt{1-\frac{x^2}{4}}}, \qquad \frac{dy}{dx} = \frac{1}{2} \times \frac{1}{x^2+1},$$

$$= \frac{1}{\sqrt{4-x^2}}. \qquad \text{When } x = 0,\ \frac{dy}{dx} = \frac{1}{2}.$$

When $x = 0$, $\dfrac{dy}{dx} = \dfrac{1}{2}$.

$\therefore$ The tangents have a common slope and a common point,
*i.e.*, a common tangent.

$$\begin{aligned} y - \left(-\tfrac{\pi}{2}\right) &= \tfrac{1}{2}(x-0), \\ 2y + \pi &= x, \\ x - 2y - \pi &= 0 \text{ is the equation of the common tangent.} \end{aligned}$$

(e) Given the quadratic equation $x^2 - x - 3 = 0$ with roots $\alpha_1$, $\alpha_2$: 5

(i) Show that $x^4 = 7x + 12$.

**Solution:**

$$\begin{aligned} x^2 &= x + 3, \\ x^4 &= x^2 + 6x + 9, \\ &= (x+3) + 6x + 9, \\ &= 7x + 12. \end{aligned}$$

(ii) Hence or otherwise find a quadratic equation with roots $\alpha_1^4$ and $\alpha_2^4$.

**Solution:** Method 1—

$$\begin{aligned}
\text{Put } y &= x^4, \ i.e., \ x = y^{1/4}, \\
y &= 7y^{1/4} + 12, \\
y^{1/4} &= \frac{y-12}{7}. \\
0 &= \left(\frac{y-12}{7}\right)^2 - \frac{y-12}{7} - 3, \\
&= y^2 - 24y + 144 - 7y + 84 - 147, \\
&= y^2 - 31y + 81.
\end{aligned}$$

So the desired equation is $x^2 - 31x + 81 = 0$.

**Solution:** Method 2—

$$\begin{aligned}
\alpha_1 + \alpha_2 &= 1, \\
\alpha_1\alpha_2 &= -3, \\
\alpha_1^4 &= 7\alpha_1 + 12, \\
\alpha_2^4 &= 7\alpha_2 + 12, \\
\alpha_1^4 + \alpha_2^4 &= 7(\alpha_1 + \alpha_2) + 24, \\
&= 7(1) + 24, \\
&= 31. \\
\alpha_1^4\alpha_2^4 &= 49\alpha_1\alpha_2 + 84(\alpha_1 + \alpha_2) + 144, \\
&= 49(-3) + 84(1) + 144, \\
&= 81. \\
\therefore \ x^2 - 31x + 81 &= 0.
\end{aligned}$$

**Solution:** Method 3—

$$\begin{aligned}
\alpha_1 + \alpha_2 &= 1, \\
\alpha_1\alpha_2 &= -3, \\
(\alpha_1 + \alpha_2)^2 &= \alpha_1^2 + 2\alpha_1\alpha_2 + \alpha_2^2 = 1, \\
\alpha_1^2 + \alpha_2^2 &= 1 - 2(-3), \\
&= 7, \\
\alpha_1^2\alpha_2^2 &= 9, \\
(\alpha_1^2 + \alpha_2^2)^2 &= \alpha_1^4 + 2\alpha_1^2\alpha_2^2 + \alpha_2^4 = 49, \\
\alpha_1^4 + \alpha_2^4 &= 49 - 2(9), \\
&= 31, \\
\alpha_1^4\alpha_2^4 &= 81, \\
\therefore \ x^2 - 31x + 81 &= 0.
\end{aligned}$$

**Solution:** Method 4—

$$\begin{aligned}
\text{Put } y &= x^4,\ i.e.,\ x = y^{1/4}, \\
\left(y^{1/4}\right)^2 - y^{1/4} - 3 &= 0, \\
y^{1/4} &= y^{1/2} - 3, \\
\left(y^{1/4}\right)^2 &= \left(y^{1/2} - 3\right)^2, \\
y^{1/2} &= y - 6y^{1/2} + 9, \\
\left(7y^{1/2}\right)^2 &= (y+9)^2, \\
49y &= y^2 + 18y + 81, \\
0 &= y^2 - 31y + 81.
\end{aligned}$$

So the desired equation is $x^2 - 31x + 81 = 0$.

**Solution:** Method 5—

$$\begin{aligned}
\alpha^2 - \alpha + \frac{1}{4} &= 3 + \frac{1}{4}, \\
\left(\alpha - \frac{1}{2}\right)^2 &= \frac{13}{4}, \\
\alpha - \frac{1}{2} &= \pm\frac{\sqrt{13}}{2}, \\
\alpha &= \frac{1 \pm \sqrt{13}}{2}, \\
\alpha^2 &= \frac{1 \pm 2\sqrt{13} + 13}{4}, \\
&= \frac{14 \pm 2\sqrt{13}}{4}, \\
&= \frac{7 \pm \sqrt{13}}{2}, \\
\alpha^4 &= \frac{49 \pm 14\sqrt{13} + 13}{4}, \\
&= \frac{62 \pm 14\sqrt{13}}{4}, \\
&= \frac{31 \pm 7\sqrt{13}}{2}, \\
\alpha_1^4 + \alpha_2^4 &= 31, \\
\alpha_1^4\alpha_2^4 &= \frac{31^2 - 49 \times 13}{4}, \\
&= 81, \\
\therefore\ x^2 - 31x + 81 &= 0.
\end{aligned}$$

QUESTION 3

(a) (i) $z^5=1$

$z_0 = \text{cis}\,0 = 1$

$z_1 = \text{cis}\frac{2\pi}{5}$ 2

$z_2 = \text{cis}\frac{4\pi}{5}$

$z_3 = \text{cis}-\frac{2\pi}{5} = \overline{z_1}$

$z_4 = \text{cis}-\frac{4\pi}{5} = \overline{z_2}$

(ii) $(z-1)(z-z_1)(z-\overline{z_1})(z-z_2)(z-\overline{z_2}) = z^5-1$ 2

$(z^2-2z\cos\frac{2\pi}{5}+1)(z^2-2z\cos\frac{4\pi}{5}+1) \equiv z^4+z^3+z^2+z+1.$

(iii) coeff of $z$

$-2\cos\frac{2\pi}{5} - 2\cos\frac{4\pi}{5} = 1.$

1

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

$\cos 2A = 2\cos^2 A - 1$

$\cos\frac{2\pi}{5} + 2\cos^2\frac{2\pi}{5} - 1 = -\frac{1}{2}.$

$4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0.$

Let $u = \cos\frac{2\pi}{5}$

$4u^2 + 2u - 1 = 0$

$(u+\frac{1}{4})^2 - \frac{1}{16} = \frac{4}{16}$

$$u = -\frac{1}{4} \pm \frac{\sqrt{5}}{4}.$$ 2

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$$ since $\frac{2\pi}{5}$ is in the first quadrant.

(b) $p(x) = A(x)Q(x) + (ax+b).$

$p(x) = (x-2)(x-3)Q(x) + (ax+b)$

$p(2) = 2a+b = 4$ ①

$p(3) = 3a+b = 9$ ② 2

② − ①

$a = 5.$

$10 + b = 4.$
$b = -6.$

remainder is $5x-6$

(c) (i) $9!$ 1

(ii) $4! \times 2 \times 3 \times 8 = 1152$

or

$2^3 \times 3! \times {}^4C_3 \times 3! = 1152$ 2

$$\frac{1152}{9!} = \frac{1}{315}$$

(d) $ax^4+bx+c=0$ — If this has a triple root then

$4ax^3+b=0$ — has a double root. So

$12ax^2=0$ — has a single root. which is,

$x=0$.

But $x=0$ is not a root of $ax^4+bx+c=0$.

(e) $\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx$

$x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta.$

$dx = 2\cos\theta\, d\theta.$

| $x$ | $\theta$ |
|---|---|
| $\sqrt{3} \rightarrow$ | $\frac{\pi}{3}$ |
| $1 \rightarrow$ | $\frac{\pi}{6}$ |

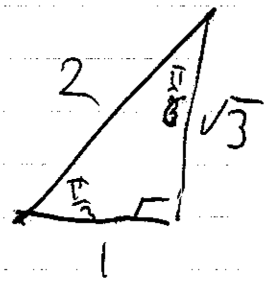
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sin^2\theta \; 2\cos\theta}{\sqrt{4-4\sin^2\theta}}\, d\theta$$

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2\theta\cos\theta}{\sqrt{1-\sin^2\theta}}\, d\theta$$

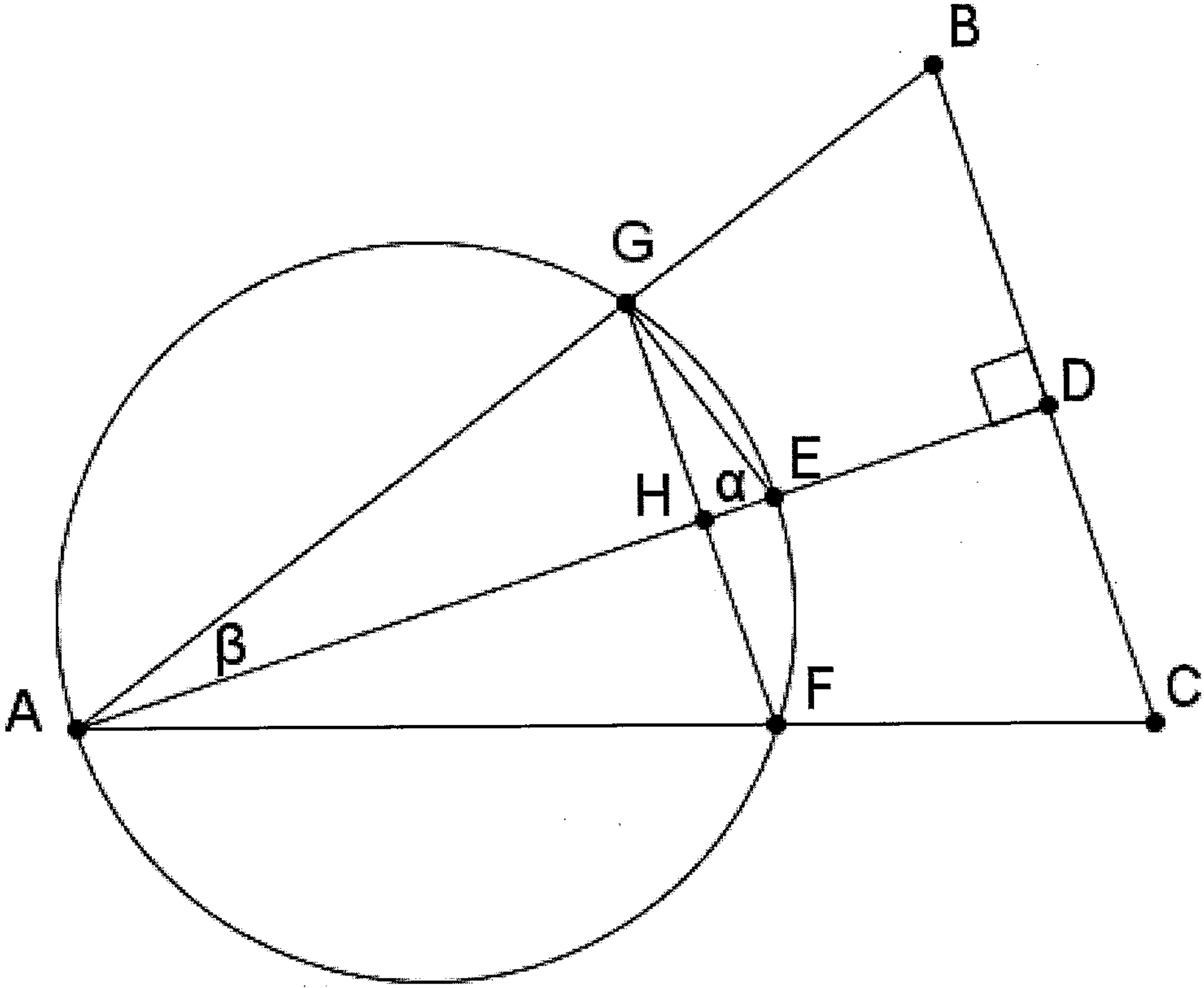
$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta\, d\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1-\cos 2\theta\, d\theta$$

$$= 2\left[\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$




$$= 2\left[\left(\frac{\pi}{3} - \frac{\sin\frac{2\pi}{3}}{2}\right) - \left(\frac{\pi}{6} - \frac{\sin\frac{\pi}{3}}{2}\right)\right]$$


B
G
D
E
H
α
β
A
F
C

$= 2\left[\frac{\pi}{3} - \frac{\pi}{6} - \sin\frac{\pi}{3}\cos\frac{\pi}{3} + \frac{\sin\frac{\pi}{3}}{2}\right]$

$= \frac{\pi}{3} - 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2}$

3

$= \frac{\pi}{3}$

(f) Join GF. Let H be the intersection of GF and AE.

Join GE

Let $\angle GEA = \alpha^\circ$ and $\angle GAE = \beta^\circ$

$\Delta AGE$ is a right angle triangle with $\angle AGE = 90^\circ$ (angle in a semi-circle)

$\therefore \alpha + \beta = 90^\circ$ ($\angle$ sum $\Delta$)

Since $\Delta ADB$ is right angle triangle $\angle ABD = \alpha^\circ$ ($\angle$ sum of a $\Delta$).

$\angle GFA = \alpha^\circ$ ($\angle$s in the same segment).

$\angle GFC = 180^\circ - \alpha^\circ$ (supplementary).

$\therefore BCFG$ is cyclic (opposite $\angle$s are supplementary)

4